Optimal Auction Design for Mixed Bidders

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Abstract

The predominant setting in classic auction theory considers bidders as *utility maximizers (UMs)*, who aim to maximize quasi-linear utility functions. Recent autobidding strategies in online advertising have sparked interest in auction design with *value maximizers (VMs)*, who aim to maximize the total value obtained. In this work, we investigate revenue-maximizing auction design for selling a single item to a mix of UMs and VMs. Crucially, we assume the UM/VM type is private information of a bidder. This shift to a multi-parameter domain complicates the design of incentive compatible mechanisms. Under this setting, we first characterize the optimal auction structure for auctions with a single bidder. We observe that the optimal auction moves gradually from a first-price auction to a Myerson auction as the probability of the bidder being a UM increases from 0 to 1. We also extend our study to multi-bidder setting and present an algorithm for deriving the optimal lookahead auction with multiple mixed types of bidders.

Introduction

In the past decades, online advertising has experienced significant success through effective auction design. The role of online advertising auctions in generating revenue for many IT companies is immense. At the same time, the scale and complexity of the online advertising market have led to the development and adoption of more efficient auction designs for the online environment.

The classical auction theory mainly builds upon the utility maximizer (UM) model; the objective of a UM is to maximize her (quasi-linear) *utility*, which is the difference between the allocated value and the payment. In the meanwhile, the growing adoption of autobidding in online advertising has motivated a new *value maximizer (VM)* paradigm (Aggarwal, Badanidiyuru, and Mehta 2019; Balseiro et al. 2021; Deng et al. 2021). Unlike the UMs, the objective of a VM is to maximize the total value she receives. VM can be used to model, for example, a company's advertising department that cannot collect the unspent budget (Lu, Xu, and Zhang 2023). Consider an ad exchange (ADX) in online ads systems, which is the auctioneer facing bids from the demand-side platforms (DSPs). Modern DSPs provide auto-bidding services for advertisers, meaning DSPs know the bidder's value and type, whereas ADX does not. In this scenario, a well-designed auction format can assist in generating more profits and facilitating better resource allocation.

Besides advertising auctions, art auctions can also involve scenarios where both UM and VM coexist. Art pieces are unique and often hold sentimental or historical value, making collectors or enthusiasts more inclined to purchase them at higher prices. Hence, they could be considered VMs in this context. However, some other bidders buy the art pieces for their potential investment value. These bidders should be modeled as UMs, resulting in the coexistence of UMs and VMs in art auctions. Note that VMs are usually more willing to spend money in an auction, which means there is a large potential for the mechanism to extract more revenue from them.

In most previous works, UMs and VMs are usually considered separately. That is, all bidders in an auction are assumed to have the same UM/VM type. However, since different advertisers usually have different bidding strategies, a practical online advertising platform will usually see a *mixture* of UMs and VMs participating in an auction simultaneously, and moreover, the auctioneer *cannot* distinguish between them.

The presence of both UMs and VMs brings additional challenges designing Incentive Compatible (IC) and Individually Rational (IR) mechanisms. Specifically, relying on the bidders who bid their own types makes them able to benefit by misreporting both of their values *and/or* UM/VM types. The problem moves to the multi-parameter from the single-parameter domain, where people have little understanding of the optimal auction design in most settings.

In this work, we investigate the revenue-maximizing optimal auction design for mixed bidders under the Bayesian setting[1]. Our main results are:

- We provide a neat characterization of IC auctions in the presence of both UMs and VMs simultaneously. We then express the revenue as a function only related to the allocation for UMs. Unlike the single-parameter environment in which the objective is linear with respect to the allocation function, our characterization and revenue expression are more complex in the mixed bidder setting.

[1]Mechanism based on known prior of bidder's types and values.

- We then apply the variation method to solve the optimal allocation function and present an algorithm to compute the optimal mechanism for a single bidder. We also provide an analytical description of the optimal mechanism when the bidder's value follows uniform distribution;
- Finally, we provide an algorithm to solve the optimal lookahead auction in the multiple mixed bidders setting.

Additional Related Works (Lv et al. 2023b) takes the first steps on the truthful auction design for mixed bidders. In their setting, bidders' types and values are private but fixed. Their mechanism can achieve a constant approximation to the optimal social welfare. Compared to their work, we consider a Bayesian setting where the bidders' types and values follow a known distribution independently. The objective function in this paper is revenue, and we can achieve optimal.

Another modeling for bidders in an online advertising platform by (Lv et al. 2023a) is UMs with return-on-investment (ROI) constraints. The typical UMs are those with low ROI constraints while VMs with high ones. The optimal auction for a single bidder with ROI constraints is characterized, but the ROI constraints are public information in their setting.

Other related literature includes works designing mechanisms for UMs and VMs separately, for which we only survey some representative ones. For UMs, Myerson did the most seminal work on auction theory and designed the optimal mechanism for selling a single item to multiple bidders whose values follow independent random distributions (Bayesian setting) (Myerson 1981). His approach can be extended to a broader scenario called single-parameter environment[2]. Other mechanism designs for UMs include position auctions(Varian 2007; Edelman, Ostrovsky, and Schwarz 2007; Chawla and Hartline 2013), multi-dimensional screening(Chawla et al. 2010; Daskalakis, Deckelbaum, and Tzamos 2015; Hart and Reny 2015), and approximate mechanisms(Hart and Nisan 2017; Jin et al. 2020). For VMs, most existing literature focuses on the efficiency of some special auction format (Deng et al. 2022, 2023; Lu, Xu, and Zhang 2023) or designing new mechanisms (Balseiro et al. 2023, 2021).

Preliminaries

We first introduce the auction model with two types of bidders. Next, we present the multi-parameter mechanism settings for the optimal auction design problem and the IR and IC constraints.

Auction Model

We consider the standard Bayesian single-item auction model. In this model, one auctioneer, also referred to as the seller, sells one item to n potential buyers, also known as bidders. Our model includes two types of bidders: UM and VM. The type of the i-th bidder, $t_i \in \{U, V\}$, represents UM and VM, respectively. The private value of the i-th bidder is v_i. The class of the i-th bidder is the pair of the type and the value, denoted as $\theta_i = (t_i, v_i)$. Let $\boldsymbol{v} = (v_i)_{i\in[n]}$, $\boldsymbol{t} = (t_i)_{i\in[n]}$, and $\boldsymbol{\theta} = (\theta_i)_{i\in[n]}$. A UM's utility is the difference between the allocated value and the payment, while a VM's utility is simply the allocated value. Each bidder, whether UM or VM, seeks to maximize their utility. We assume that the distribution of each bidder's θ_i is identical and known. Each bidder is a UM with probability q and a VM with probability $1-q$, independently. If a bidder's type is UM, her value v_i follows the distribution $\mathcal{F}^U$, with probability density function (pdf) $f^U(\cdot)$, independently and identically. If the bidder is a VM, her value follows the distribution $\mathcal{F}^V$, with pdf $f^V(\cdot)$, independently and identically. In this paper, we assume $\mathcal{F}^U$ is regular[3], and there is no assumption on $\mathcal{F}^V$.

Multi-Parameter Mechanisms

We consider a multi-parameter mechanism, where bidders submit bids that include both their values and their types. Let $b_i = (\hat{t}_i, \hat{v}_i)$ represent the bid of the i-th bidder, and $\boldsymbol{b} = (b_i)_{i\in[n]}$ represent the bid profile. After receiving the bids, the mechanism determines $\boldsymbol{x}(\boldsymbol{b}) = (x_i(\boldsymbol{b}))_{i\in[n]}$ and $\boldsymbol{p}(\boldsymbol{b}) = (p_i(\boldsymbol{b}))_{i\in[n]}$, representing the allocation and payments, respectively. Let $u_i(\boldsymbol{b})$ denote the utility of the i-th bidder under the bid profile $\boldsymbol{b}$. According to the definition of the two types of bidders, we have

$$u_i(\boldsymbol{b}) = \begin{cases} x_i(\boldsymbol{b})v_i - p_i(\boldsymbol{b}) & \text{if } t_i = U, \\ x_i(\boldsymbol{b}) \cdot v_i & \text{if } t_i = V. \end{cases}$$

Denote $\mathcal{M} = (\boldsymbol{x}, \boldsymbol{p})$, which is the pair of allocation and payment functions. Since there is only one item to be sold, $\sum_{i=1}^n x_i(\boldsymbol{b}) \leq 1, \forall \boldsymbol{b}$. In this paper, we focus on designing a mechanism that satisfies the following IR and IC constraints.

(Ex-post) IR Constraints If the i-th bidder bids truthfully, the payment should not be larger than her value, whatever the other bidders bid.

$$\forall i, \boldsymbol{b}_{-i}, \quad p_i(\theta_i, \boldsymbol{b}_{-i}) \leq v_i \cdot x_i(\theta_i, \boldsymbol{b}_{-i}),$$

where $\boldsymbol{b}_{-i}$ is the bids of all bidders except the i-th bidder. Note that although the utility of the VMs is not related to the payments, the IR constraints require their payment to be less than their value, which prevents us from extracting a large amount of revenue from them.

(Ex-ante) IC Constraints We consider the Bayesian Incentive Compatibility (BIC) constraints in our model. That is,

$$\forall i, \hat{b}, \quad \mathbb{E}_{\boldsymbol{\theta}_{-i}}[u_i(\hat{b}, \boldsymbol{\theta}_{-i})] \leq \mathbb{E}_{\boldsymbol{\theta}_{-i}}[u_i((t_i, v_i), \boldsymbol{\theta}_{-i})],$$

where $\boldsymbol{\theta}_{-i}$ is the classes of all bidders except the i-th bidder.

[2]We provide a brief introduction about the single parameter environment in full version.

[3]A distribution $\mathcal{F}$ is regular if the virtual value function $\varphi(v) = v - \frac{1-\mathcal{F}(v)}{f(v)}$ is monotone increasing where $f(\cdot)$ is the pdf.

Revenue Maximization

The seller aims to maximize the revenue, which is the sum of payments from all bidders. We denote the revenue by $\mathsf{REV}(\mathcal{M}) = \sum_{i=1}^{n} p_i(\boldsymbol{b})$. The problem can be viewed as the following optimization problem

$$\max_{\boldsymbol{x}(\cdot),\boldsymbol{p}(\cdot)} \quad \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i=1}^{n} p_i(\boldsymbol{\theta})] \qquad \text{(REV OPT)}$$

$$\begin{aligned} s.t. \quad & \sum_{i=1}^{n} x_i(\boldsymbol{\theta}) \leq 1, \quad \forall \boldsymbol{\theta} \\ & \boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot) \text{ satisfy IR and IC constraints,} \\ & x_i(\boldsymbol{\theta}), p_i(\boldsymbol{\theta}) \geq 0, \quad \forall \boldsymbol{\theta}. \end{aligned}$$

Characterization of the IC Auctions Structure

In this section, we characterize the Incentive Compatible (IC) auctions with mixed bidder types. More precisely, given any fixed interim allocation for UMs that is monotonically increasing and continuous, we characterize the interim payments for both bidder types and the interim allocation for VMs for a revenue-maximizing auction. This characterization is more intricate compared to the single-parameter auctions due to the added complexity of two-dimensional parameters affecting the IC constraints. We note that an IR and IC single-parameter mechanism that allows bidders to only bid their values and not their classes cannot achieve any reasonable revenue for mixed bidders, particularly when bidders are more likely to be VMs. (Refer to the full version for a detailed discussion.)

Denote

$$\overline{x}_i^U(v_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[x_i((U, v_i), \boldsymbol{\theta}_{-i})],$$
$$\overline{p}_i^U(v_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[p_i((U, v_i), \boldsymbol{\theta}_{-i})],$$

as the interim allocation and payment for the i-th bidder respectively if her type is UM and value is v_i. Similarly, let

$$\overline{x}_i^V(v_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[x_i((V, v_i), \boldsymbol{\theta}_{-i})],$$
$$\overline{p}_i^V(v_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[p_i((V, v_i), \boldsymbol{\theta}_{-i})],$$

be the interim allocation and payment for a VM with value v_i respectively. Denote $\overline{v}^U = \sup\{\text{supp}(\mathcal{F}^U)\}$[4]. We have the following characterization.

Theorem 1. *Given $\overline{x}_i^U(\cdot)$ which is monotonically increasing and continuous. Denote*

$$g_i(v_i) = \begin{cases} \frac{\overline{p}_i^U(v_i)}{\overline{x}_i^U(v_i)} & \overline{x}_i^U(v_i) > 0; \\ v_i & \overline{x}_i^U(v_i) = 0, \end{cases}$$

as the expected payment per unit of UMs with value v_i. The following payment and allocation rules are revenue-optimal under the IR and IC constraints:

- $\overline{p}_i^U(v_i) = v_i \cdot \overline{x}_i^U(v_i) - \int_0^{v_i} \overline{x}_i^U(u)\mathrm{d}u;$
- $\overline{x}_i^V(v_i) = \begin{cases} \overline{x}_i^U(g_i^{-1}(v_i)) & \text{if } 0 \leq v_i \leq g_i(\overline{v}^U), \\ \min\{\frac{\int_0^{\overline{v}^U} \overline{x}^U(u)\mathrm{d}u}{\overline{v}^U - v_i}, 1\} & \text{if } g_i(\overline{v}^U) < v_i \leq \overline{v}^U, \\ 1 & \text{if } v_i > \overline{v}^U; \end{cases}$
- $\overline{p}_i^V(v_i) = v_i \cdot \overline{x}_i^V(v_i).$

This characterization theorem allows us to transform the optimization (REV OPT) into a simpler optimization problem solely for the interim allocation for UMs.

The remaining of this section is dedicated to the proof of this theorem. We start by characterizing the IC payment rules.

Payment Rules

We follow a similar idea as (Myerson 1981) to first characterize the unique IR and IC payment rule for any given allocation rule. This structural result helps us express the auction revenue as a function of the allocation rule.

Since the bidders can deviate in two dimensions (types and value), there are four groups of IC constraints: **UM-UM**, **VM-VM**, **UM-VM**, and **VM-UM**, where **A-B** means a bidder of type A cannot benefit from bidding type B and a possibly different value.

UM-UM: We first focus on the constraints that any UM i cannot benefit from bidding the true type $t_i = U$, but an untruthful $\hat{v}_i$ rather than the true value v_i. Formally,

$$\forall \hat{v}, \quad v_i \cdot \overline{x}_i^U(\hat{v}) - \overline{p}_i^U(\hat{v}) \leq v_i \cdot \overline{x}_i^U(v_i) - \overline{p}_i^U(v_i).$$

Such constraints are exactly the same as those in the single-parameter auction setting. Thus, by Myerson's Lemma we have:

Lemma 1 (Myerson 1981)**.** *For given $\overline{x}_i^U(\cdot)$, the only payment satisfying the IC constraints for the UMs is:*

$$\overline{p}_i^U(v_i) = v_i \cdot \overline{x}_i^U(v_i) - \int_0^{v_i} \overline{x}_i^U(u)\mathrm{d}u.$$

Moreover, the expected utility of a UM with value v_i is:

$$\forall v_i, \quad u_i^U(v_i) = v_i \cdot \overline{x}_i^U(v_i) - \overline{p}_i^U(v_i) = \int_0^{v_i} \overline{x}_i^U(u)\mathrm{d}u.$$

The payment function here is called Myerson's payment for UMs. Recall that we define

$$g_i(v_i) = \frac{\overline{p}_i^U(v_i)}{\overline{x}_i^U(v_i)} = v_i - \frac{\int_0^{v_i} \overline{x}_i^U(u)\mathrm{d}u}{\overline{x}_i^U(v_i)},$$

as the expected payment per unit of UMs with value v_i, then Myerson's payment can be expressed as:

$$p_i((U, v_i), \boldsymbol{\theta}_{-i}) = x_i((U, v_i), \boldsymbol{\theta}_{-i}) \cdot g_i(v_i).$$

Payment for VMs. We can also show that the optimal payment function for VMs must be the first price payment. That is, the payment of the winner equals to the value she bids times her allocation. Indeed, it can be easily verified that for any IR and IC mechanism, increasing $\overline{p}_i^V(v_i)$ to $v_i\overline{x}_i^V(v_i)$ for a VM will not violate the IR and IC constraints. Thus, a revenue-optimal mechanism should always have the first price payment for VMs.

Allocation Rules for VMs

Next, we discuss $\overline{x}_i^V(\cdot)$ satisfying the IC constraints for given $\overline{x}_i^U(\cdot)$. Under Myerson's and the first price payment for UMs and VMs respectively derived in the last subsection, we consider the other three groups of IC constraints:

[4]For a distribution $\mathcal{F}$ whose pdf is $f(\cdot)$, $\text{supp}(\mathcal{F}) = \{v|f(v) > 0\}$.

VM-VM: The interim allocation for the VMs should be monotone:

$$\forall \hat{v} \le v_i, \quad \overline{x}_i^V(\hat{v}) \le \overline{x}_i^V(v_i).$$

UM-VM: A UM will not misreport her type as VM with a higher value, since her utility will be non-positive then. As for misreporting a lower value, the IC constraints require:

$$\forall \hat{v} \le v_i, \quad u_i^U(v_i) = \int_0^{v_i} \overline{x}_i^U(u)\mathrm{d}u \ge (v_i - \hat{v}) \cdot \overline{x}_i^V(\hat{v}).$$

VM-UM: A VM would only misreport as a UM with $\hat{v}$ such that $v_i \ge g_i(\hat{v})$ (IR constraints for VM). Then we have the IC constraints for such $\hat{v}$:

$$\forall \hat{v}, \text{ s.t. } g_i(\hat{v}) \le v, \quad \overline{x}_i^U(\hat{v}) \le \overline{x}_i^V(v_i).$$

From these constraints, we can see that the IC condition for $\overline{x}_i^V$ is closely related to $g_i(\cdot)$. The following lemma characterizes several basic properties of $g_i(\cdot)$:

Lemma 2. *The $g_i(\cdot)$ function has the following properties:*

- *$g_i(\cdot)$ is monotone increasing;*
- *$\forall v_1, v_2$, we have $\overline{x}_i^U(v_1) = \overline{x}_i^U(v_2)$ if and only if $g_i(v_1) = g_i(v_2)$;*
- *If $\overline{x}_i^U(\cdot)$ is continuous, then $g_i(\cdot)$ is also continuous;*
- *Given any fixed $g_i(\cdot)$, we have*

$$\overline{x}_i^U(v) = C \cdot \exp\left(\int_0^v \frac{g_i'(u)}{u - g_i(u)}\mathrm{d}u\right),$$

for some constant C.

The last property in Lemma 2 suggests that to optimize $\overline{x}_i^U(\cdot)$, we can instead focus on optimizing $g_i(\cdot)$. The proof of this lemma can be found in the full version.

When $\overline{x}_i^U(\cdot)$ is given, the following lemma characterizes a necessary condition of IC $\overline{x}_i^V(\cdot)$.

Lemma 3 (Necessary condition of IC $\overline{x}_i^V(\cdot)$). *For any given $\overline{x}_i^U(\cdot)$, suppose $\overline{x}_i^V(\cdot)$ satisfies the IC constraints, then for any $v_i, \underline{v}, \overline{v}$, such that $g_i(\underline{v}) \le v_i \le g_i(\overline{v})$, we have*

$$\overline{x}_i^U(\underline{v}) \le \overline{x}_i^V(v_i) \le \overline{x}_i^U(\overline{v}).$$

Specially, if there exists v^*, such that $g_i(v^*) = v_i$, then $\overline{x}_i^V(v_i) = \overline{x}_i^U(v^*)$.

Proof: According to the **VM-UM** IC constraints, we have $\overline{x}_i^V(v_i) \ge \overline{x}_i^U(\underline{v})$, which is the first inequality.

For the second inequality, we have

$$\begin{aligned}
\overline{x}_i^V(v_i) &\le \frac{u_i^U(\overline{v})}{\overline{v} - v_i} && \text{(UM-VM IC constraints)} \\
&\le \frac{u_i^U(\overline{v})}{\overline{v} - g_i(\overline{v})} && (v_i \le g_i(\overline{v})) \\
&= \frac{\overline{x}_i^U(\overline{v}) \cdot \overline{v} - \overline{p}_i^U(\overline{v})}{\overline{v} - \frac{\overline{p}_i^U(\overline{v})}{\overline{x}_i^U(\overline{v})}} = \overline{x}_i^U(\overline{v}).
\end{aligned}$$

□

With the help of Lemma 3 and mild assumptions on $\overline{x}_i^U(\cdot)$, we can derive a necessary and sufficient condition of $\overline{x}_i^V(\cdot)$.

Lemma 4 (Necessary and sufficient condition of IC $\overline{x}_i^V(\cdot)$ for continuous $\overline{x}_i^U(\cdot)$). *When $\overline{x}_i^U(\cdot)$ is continuous, the necessary and sufficient condition for IC $\overline{x}_i^V(\cdot)$ is that:*

1. *$\overline{x}_i^V(\cdot)$ is monotonically increasing;*
2. *$\forall 0 \le v_i \le g_i(\overline{v}^U)$, $\overline{x}_i^V(v_i) = \overline{x}_i^U(g_i^{-1}(v_i))$;*
3. *$\forall g_i(\overline{v}^U) < v_i \le \overline{v}^U$, $\overline{x}^V(v_i) \le \frac{\int_0^{\overline{v}^U} \overline{x}^U(u)\mathrm{d}u}{\overline{v}^U - v_i}$.*

Proof: We first show that the allocation is well-defined. Specifically, by the second property of Lemma 2, we know that $\overline{x}_i^U(g_i^{-1}(v_i))$ exists and is unique, even when $g_i^{-1}(v_i)$ is a set of values.

For necessity, $\overline{x}_i^V(\cdot)$ should be monotone increasing from the **VM-VM** IC constraints. Condition (2) holds by Lemma 3 and the continuity of $\overline{x}_i^U$, and condition (3) holds due to the **UM-VM** IC constraints for UM with value $\overline{v}^U$.

For the sufficiency, the **VM-VM** and **VM-UM** constraints both hold for the monotonicity of $\overline{x}_i^U(\cdot), \overline{x}_i^V(\cdot)$ and $g_i(\cdot)$. It suffices to show that the **UM-VM** constraints hold. We first consider the UMs with value $0 \le v_i \le g_i(\overline{v}^U)$. They will not deviate to a VM with a value larger than $g_i(\overline{v}^U)$. For any value $\hat{v} \le g_i(\overline{v}^U)$ we have

$$\begin{aligned}
(v_i - \hat{v}) \cdot \overline{x}_i^V(\hat{v}) &= (v_i - g_i(g_i^{-1}(\hat{v}))) \cdot \overline{x}_i^U(g_i^{-1}(\hat{v})) \\
&= v_i \cdot \overline{x}_i^U(g_i^{-1}(\hat{v})) - \overline{p}_i^U(g_i^{-1}(\hat{v})) \le u_i^U(v_i).
\end{aligned}$$

The second equality is because $g_i(v_i) \cdot \overline{x}_i^U(v_i) = \overline{p}_i^U(v_i)$ for any v_i. The third inequality is from the **UM-UM** IC constraints which can be guaranteed only by Myerson's payments.

If $v_i > g_i(\overline{v}^U)$, then for any $\hat{v} \le v_i$ and $\hat{v} \in [g(\overline{v}^U), \overline{v}^U]$ we have:

$$\begin{aligned}
(v_i - \hat{v}) \cdot \overline{x}^V(\hat{v}) &\le (v_i - \hat{v}) \frac{\int_0^{\overline{v}^U} \overline{x}^U(u)\mathrm{d}u}{\overline{v}^U - \hat{v}} \\
&\le \frac{v_i - g(\overline{v}^U)}{\overline{v}^U - g(\overline{v}^U)} \cdot \int_0^{\overline{v}^U} \overline{x}^U(u)\mathrm{d}u \\
&= v_i \cdot \overline{x}^U(\overline{v}^U) - \overline{p}^U(\overline{v}^U) \\
&\le u^U(v_i).
\end{aligned}$$

The second inequality holds as $\hat{v} \ge g(\overline{v}^U)$, and the last inequality holds as the UM-UM IC constraints which is guaranteed by the Myerson's payment. □

Lemma 4 reveals an important connection between UM and VM in an IR and IC auction: *the interim allocation and payment must be identical for a UM with value v_i and a VM with value $g_i(v_i)$.*

With Lemma 4 in hand, the proof of the Theorem 1 is straightforward. When $\overline{x}_i^U(\cdot)$ is given, the revenue is monotone increasing with respect to the value of $\overline{x}_i^V(\cdot)$. Therefore, to maximize the revenue as stated in Theorem 1, the third property of Lemma 4 should be utilized to take the maximum.

Optimal Auction Design for a Single Bidder

We now solve the revenue-maximization problem for a single bidder. This can be achieved by first expressing the revenue objective in terms of $\overline{x}_i^U(\cdot)$, then applying a variational method to optimize it. We also provide a numerical algorithm to solve the ordinary differential equation (ODE) arising from the variational method, and demonstrate that our theory aligns with the results in the literature in two extreme scenarios. Additionally, we derive the optimal mechanism in closed form for the case where the value distributions follow a uniform distribution, which is detailed in the full version.

Problem Reformulation

We begin by expressing the revenue objective with respect to $\overline{x}_i^U(\cdot)$ in the multi-bidder case:

$$\begin{aligned}\mathsf{REV}(\mathcal{M}) &= \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i=1}^n p_i(\boldsymbol{\theta})] \\ &= \sum_{i=1}^n \int_0^{+\infty} q\cdot \overline{p}_i^U(v) f^U(v) + (1-q)\cdot \overline{p}_i^V(v) f^V(v) \mathrm{d}v \\ &= \sum_{i=1}^n \int_0^{+\infty} q\cdot \overline{x}_i^U(v)\varphi^U(v) f^U(v) \\ &\quad + (1-q)\cdot \overline{x}_i^V(v)\cdot v\cdot f^V(v)\mathrm{d}v,\end{aligned}$$

where the second equality is from the revenue equivalence and $\overline{p}_i^V(v) = \overline{x}_i^V(v)\cdot v$. We define

$$h_i^U(v) = q\cdot\varphi^U(v) f^U(v), h_i^V(v) = (1-q)\cdot v\cdot f^V(v).$$

Then the revenue can be expressed as

$$\sum_{i=1}^n \int_0^{+\infty} \left(\overline{x}_i^U(v) h_i^U(v) + \overline{x}_i^V(v) h_i^V(v)\right) \mathrm{d}v.$$

Now we turn to the single-bidder case. Since there is only one bidder, we omit the subscript i. Note that to maximize the revenue, we should always allocate the item to the UM when she has the largest value in the support. Denote $\overline{v} = \inf_v\{\overline{x}^U(v) = 1\}$. The optimal $\overline{x}^V(\cdot)$ in Theorem 1 can then be expressed as:

$$\overline{x}^V(v) = \begin{cases} \overline{x}^U(g^{-1}(v)) & \text{if } 0\le v\le g(\overline{v}), \\ 1 & \text{if } v > g(\overline{v}).\end{cases}$$

Thus, the second term satisfies

$$\begin{aligned}&\int_0^{+\infty} \overline{x}^V(v) h^V(v)\mathrm{d}v \\ &= \int_0^{g(\overline{v})} \overline{x}^U(g^{-1}(v)) h^V(v)\mathrm{d}v + \int_{g(\overline{v})}^{+\infty} h^V(v)\mathrm{d}v \\ &= \int_0^{\overline{v}} \overline{x}^U(v) g'(v) h^V(g(v))\mathrm{d}v + \int_{g(\overline{v})}^{+\infty} h^V(v)\mathrm{d}v,\end{aligned}$$

where the last equality holds as we substitute v by $g^{-1}(v)$ in the first integral. Thus, the revenue can be expressed as

$$\begin{aligned}\mathsf{REV}(\mathcal{M}) = &\int_0^{\overline{v}} \left(\overline{x}^U(v) h^U(v) + \overline{x}^U(v) g'(v) h^V(g(v))\right) \mathrm{d}v \\ &+ \int_{\overline{v}}^{+\infty} h^U(v)\mathrm{d}v + \int_{g(\overline{v})}^{+\infty} h^V(v)\mathrm{d}v. \quad (1)\end{aligned}$$

Optimizing $\overline{x}^U(\cdot)$

Applying Theorem 1 to Eqn. (1) and denote $y(v) = \int_0^v \overline{x}^U(u)\mathrm{d}u$, we have $\overline{x}^U(v) = y'(v)$ and $g(v) = v - \frac{y(v)}{y'(v)}$, which means $\overline{x}^{U\prime}(v) = y''(v)$ and $g'(v) = \frac{y(v)y''(v)}{(y'(v)^2)}$. The revenue becomes

$$\begin{aligned}\mathsf{REV}(\mathcal{M}) = &\int_0^{\overline{v}} \left(y'\cdot h^U(v) + \frac{y\cdot y''}{y'} h^V\left(v - \frac{y}{y'}\right)\right)\mathrm{d}v \\ &+ \int_{\overline{v}}^{+\infty} h^U(v)\mathrm{d}v + \int_{g(\overline{v})}^{+\infty} h^V(v)\mathrm{d}v. \quad (2)\end{aligned}$$

If we fix $y(\overline{v})$ and let $y'(\overline{v}) = \overline{x}^U(\overline{v}) = 1$ to be the initial value, then $g(\overline{v}) = \overline{v} - y(\overline{v})$ is also fixed. This makes the last two integrals in the revenue as constants, and our target is now to maximize the first integral. Define

$$f(v, y, y', y'') = \frac{y\cdot y''}{y'} h^V(v - \frac{y}{y'}) + y'\cdot h^U(v).$$

The optimization problem becomes

$$\begin{aligned}\max \quad & \int_0^{\overline{v}} f(v, y, y', y'')\mathrm{d}v \quad (3) \\ s.t. \quad & y'(v) \text{ is monotone increasing,} \\ & 0\le y'(v)\le 1 \quad \forall v, \\ & y'(0) = 0, y'(\overline{v}) = 1.\end{aligned}$$

Applying the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}v}\frac{\partial f}{\partial y'} + \frac{\mathrm{d}^2}{\mathrm{d}v^2}\frac{\partial f}{\partial y''} = 0, \quad (4)$$

we have:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{y''}{y'}\left(h^V\left(v - \frac{y}{y'}\right) - \frac{y}{y'}\cdot h^{V\prime}\left(v - \frac{y}{y'}\right)\right), \\ \frac{\partial f}{\partial y'} &= h^U(v) - \frac{y\cdot y''}{y'}\left(h^V\left(v - \frac{y}{y'}\right) - \frac{y}{y'}\cdot h^{V\prime}\left(v - \frac{y}{y'}\right)\right), \\ \frac{\partial f}{\partial y''} &= \frac{y}{y'}\cdot h^V\left(v - \frac{y}{y'}\right).\end{aligned}$$

Substituting them into Eqn. (4) leads to

$$\forall 0\le v\le\overline{v}, \quad \frac{y''}{y'}\cdot h^V\left(v - \frac{y}{y'}\right) = h^{U\prime}(v). \quad (5)$$

This is a second-order ODE. With the initial value provided in the optimization (3), the ODE has a unique solution. The parameter $\overline{v}$ is yet to be determined and is associated with the revenue. By solving ODE (5), we can determine $y(\cdot)$. Subsequently, substituting this into Eqn. (2) gives us $\mathsf{REV}(\mathcal{M})$ as a function of $\overline{v}$. Finally, we can search all possible values of $\overline{v}$ to find the one with the optimal revenue.

While a general second-order ODE may be difficult to solve, in the following we present a numerical algorithm 1 to obtain the optimal auction. We first discretize the integral interval with gap ε. If $y(v)$ and $y'(v)$ are known, we invoke $\mathcal{RK}(v) = (y(v+\varepsilon), y'(v+\varepsilon), y''(v+\varepsilon))$ as an extrapolation method oracle for the solution of ODE (5). It can be

Algorithm 1: Find the optimal allocation for a single bidder

$\overline{v}^U \leftarrow \sup\{\mathrm{supp}(\mathcal{F}^U)\}, \overline{v}^V \leftarrow \sup\{\mathrm{supp}(\mathcal{F}^V)\}$
for all possible parameter C **do**
  $\mathsf{REV}(C) \leftarrow 0, y(\varepsilon) \leftarrow \varepsilon \cdot C, y'(\varepsilon) \leftarrow C$
  for $j = 2\varepsilon, 3\varepsilon, \ldots, \overline{v}^U$ **do**
    $(y(j), y'(j), y''(j)) \leftarrow \mathcal{RK}(j - \varepsilon)$
    $\mathsf{REV}(C) += \varepsilon \cdot \Delta\mathsf{REV}(j)$
    if $y'(j) > 1$ **then**
      break
    end if
  end for
  $\mathsf{REV}(C) += \int_j^{\overline{v}^U} h^U(v)\mathrm{d}v + \int_{j-\frac{y(j)}{y'(j)}}^{\overline{v}^V} h^V(v)\mathrm{d}v$
end for
return the C^* with the highest $\mathsf{REV}(C)$

implemented by any numerical ODE algorithm, such as the Runge-Kutta methods. We define

$$\Delta\mathsf{REV}(v) = y'(v) \cdot h^U(v) + \frac{y(v)y''(v)}{y'(v)} h^V\left(v - \frac{y(v)}{y'(v)}\right),$$

which is used to compute the revenue in our numerical algorithm. Different from the argument above, we choose $y'(\varepsilon)$ as the parameter to be optimized for convenience.

Two Extreme Scenarios

We validate our solution by solving the optimal auction in two extreme cases, $q = 1$ and $q = 0$, meaning the bidder is of type UM or VM with probability 1. We will demonstrate that these solutions precisely correspond to the optimal auction formats established in the literature.

First, rather than solving the ODE (5) directly, we can utilize the fact that $y''/y' = g'(v)/(v - g(v))$ as derived by definition, and transform ODE (5) into:

$$\forall 0 \le v \le \overline{v}, \quad \frac{g'(v)h^V(g(v))}{v - g(v)} = h^{U'}(v). \tag{6}$$

This is a first-order ODE. Once $g(v)$ is solved according to ODE (6), we can obtain $\overline{x}^U(\cdot)$ by the last property of Lemma 2:

$$\overline{x}^U(v) = C(\overline{v}) \cdot \exp\left(\int_0^v \frac{g'(u)}{u - g(u)}\mathrm{d}u\right), \tag{7}$$

where $C(\overline{v})$ is a constant determined by the initial condition $\overline{v}$. By inserting this into Eqn. (2) we can derive $\mathsf{REV}(\mathcal{M})$ as a function of $\overline{v}$. This allows us to search all possible $\overline{v}$ values to identify the $\overline{v}$ that maximizes our revenue.

When $q = 0$, we have $h^U(v) \equiv 0, h^{U'} = 0$. Thus, $g'(v) = 0$ and $g(v) \equiv 0$ from ODE (6). By substituting it into Eqn. (7) we have $\overline{x}^U(v) = \overline{x}^V(v) \equiv 1$, which is the first price auction.

When $q = 1$, we have $h^V(v) \equiv 0$. According to ODE (6), $g(v) = v, \forall 0 \le v \le \overline{v}$, which leads to

$$\overline{x}^U(v) = \begin{cases} 0 & 0 \le v < \overline{v}, \\ 1 & v \ge \overline{v}. \end{cases}$$

This is a posted price mechanism which is exactly Myerson's auction for a single bidder (after searching for the revenue-maximizing $\overline{v}$).

Optimal Auction Design for Multiple Bidders

We now turn to the general multi-bidder case. In this section, we first show the necessary and sufficient conditions for the feasibility of an interim allocation in the mixed bidder setting. Nonetheless, these conditions are too intricate to optimize $\overline{x}_i^U(\cdot)$. We then introduce the concept of the lookahead auction for mixed bidders, simplifying the characterization of interim feasibility. To determine the revenue-optimal lookahead auction, we apply the variational method with interim feasibility constraints to Equation (1).

Interim Feasibility and Lookahead Auctions

For the single-bidder scenario, the constraints on $\overline{x}_i^U(\cdot)$ are simply $0 \le \overline{x}_i^U(v) \le 1$ for any v. In the multi-bidder case, we must also consider the implementability of an interim allocation. We first provide the following necessary and sufficient conditions for interim feasibility which is known as Border's theorem :

Proposition 1 **((Border 1991)).** *Denote the set of all possible bidder's classes as T. A set of interim allocation $\{\overline{x}_i^U(\cdot), \overline{x}_i^V(\cdot)\}_{i\in[n]}$ is implementable if and only if $\forall S_1, S_2, \ldots, S_n \subseteq T$ we have:*

$$\sum_{i=1}^n \mathbb{E}[\overline{x}_i^{t_i}(v_i)|\theta_i \in S_i] \cdot \Pr(\theta_i \in S_i) \le \Pr_{\boldsymbol{\theta}}(\exists i \in [n] : \theta_i \in S_i).$$

The constraints in Border's theorem are overly complex for optimizing $\overline{x}_i^U(\cdot)$ using the variational approach. As noted in Lemma 4, the interim allocations and payments are identical for a UM with value v and a VM with value $g_i(v)$. This insight leads us to introduce the concept of the lookahead auction in our mixed bidder setting as follows:

Definition 1 **(Lookahead Auction).** *For bidder i with value v_i, define the score as $g_i(v_i)$ if this bidder is a UM, and v_i if a VM. A lookahead auction for mixed bidders satisfies:*

- *Symmetry: $\forall i, j$ and $t \in U, V, \overline{x}_i^t(v) \equiv \overline{x}_j^t(v)$;*
- *Allocation: If the item is allocated, it goes to the bidder with the highest score.*

Since we only consider symmetric mechanisms, we omit the subscript i. The following lemma characterizes the interim feasibility of $\overline{x}^U(\cdot)$ for a lookahead auction.

Lemma 5 **(Interim feasibility inequality).** *An interim allocation $(\overline{x}^U(\cdot), \overline{x}^V(\cdot))$ of a lookahead auction is implementable if and only if for all $\theta_i = (t, v)$, we have $\overline{x}^t(v) \le \Pr[i \text{ has the highest score}|\theta_i]$.*

The proof is straightforward and we put it in the full version.

Characterization of the Optimal Auction

For a single bidder, selling to a UM with positive virtual value is profitable without reducing revenue from VMs. Thus, $\overline{x}^U(v)$ can reach 1 for sufficiently large v. However, in a lookahead auction, even a UM with value $\overline{v}^U$ may lose to a VM with a value exceeding $g(\overline{v}^U)$. Lemma 5 implies that the optimal implementable allocation function for VM when $v \in [g(\overline{v}^U), \overline{v}^U]$ is:

$$\overline{x}^{V*}(v) = \min\left\{\left(q + (1-q)\cdot\mathcal{F}^V(v)\right)^{n-1}, \frac{\int_0^{\overline{v}^U}\overline{x}^U(u)\mathrm{d}u}{\overline{v}^U - v}\right\}. \quad (8)$$

For $v \in [\overline{v}^U, \overline{v}^V]$, we set $\overline{x}^V(v)$ to maximize revenue while satisfying the interim feasibility inequality. Let $y(v) = \int_0^v \overline{x}^U(u)\mathrm{d}u$. The revenue is then:

$$\mathsf{REV}(\mathcal{M}) = n\cdot\left(\int_0^{\overline{v}^U} y'\cdot h^U(v) + \frac{y\cdot y''}{y'}h^V\left(v - \frac{y}{y'}\right)\mathrm{d}v + \int_{g(\overline{v}^U)}^{\overline{v}^V}\overline{x}^V(v)\cdot h^V(v)\mathrm{d}v\right).$$

Treating $y(\overline{v}^U)$ and $y'(\overline{v}^U)$ as initial values, $g(\overline{v}^U)$ becomes fixed, making the second integral constant. We focus on optimizing the first integral. Using the definition from the previous section, the optimization problem becomes:

$$\max \int_0^{\overline{v}^U} f(v, y, y', y'')\mathrm{d}v \quad (9)$$

$$\begin{aligned} s.t. \quad & y'(v) \text{is monotone increasing,} \\ & y'(v) \leq (q\cdot\mathcal{F}^U(v_i) + (1-q)\cdot\mathcal{F}^V(v - \tfrac{y}{y'}))^{n-1} \quad \forall v, \\ & y'(v) \geq 0 \quad \forall v, \\ & y'(0) = 0. \end{aligned}$$

By complementary slackness, the Euler-Lagrange equation (4) is violated under the optimal solution only when the interim feasibility constraint binds. Let $y_1(v)$ be the solution to ODE (4), and $y_2(v)$ be the solution to the ODE derived from the interim feasibility constraints:

$$y'(v) = (q\cdot\mathcal{F}^U(v) + (1-q)\cdot\mathcal{F}^V(v - \frac{y}{y'}))^{n-1}. \quad (10)$$

The optimal $y(v)$ is then determined by the function with the smaller derivative between $y_1(v)$ and $y_2(v)$.

The Euler-Lagrange equation (5) is a second-order ODE, but problem (9) only provides one initial value, $y'(0) = 0$. This introduces a parameter C that affects revenue. For a fixed C, we can determine the optimal $y(\cdot)$ and corresponding $\mathsf{REV}(\mathcal{M})$. The optimal auction can then be found by searching all possible C values and selecting the one that maximizes revenue.

In summary, Algorithm 2 numerically solves the revenue-optimal lookahead auction for multiple bidders. We discretize the integral interval with step size ε. Given $y(v)$ and $y'(v)$, we use $\mathcal{RK}_1(v) = (y(v+\varepsilon), y'(v+\varepsilon), y''(v+\varepsilon))$ as an extrapolation oracle for ODE (5)'s solution. Similarly, $\mathcal{RK}_2(v)$ serves as an oracle for ODE (10)'s solution. $\Delta\mathsf{REV}(v)$ is defined in the previous section with $y'(\varepsilon)$ as the parameter to be determined.

Figure 1 illustrates the optimal allocation for both bidder types when $n = 2$, $q = 0.4$, and both $\mathcal{F}^U$ and $\mathcal{F}^V$ follow uniform distributions on $[0, 1]$. The green line represents the interim feasibility constraints, showing the probability of a score v being highest among all bidders. In the first phase, the Euler-Lagrange equation binds for both bidder types, with the yellow line below the green line, indicating possible non-allocation even with the highest score. The second phase begins when these lines coincide, binding the interim feasibility inequality and ensuring allocation for the highest score in this range. For VMs, the third phase's optimal allocation is determined by Eqn. (8).

Algorithm 2: Search optimal allocation for multiple bidders

$\overline{v}^U \leftarrow \sup\{\mathrm{supp}(\mathcal{F}^U)\}, \overline{v}^V \leftarrow \sup\{\mathrm{supp}(\mathcal{F}^V)\}$
for all possible parameter C **do**
  $\mathsf{REV}(C) \leftarrow 0, y(\varepsilon) \leftarrow \varepsilon\cdot C, y'(\varepsilon) \leftarrow C$
  for $j = 2\varepsilon, 3\varepsilon, \ldots, \overline{v}^U$ **do**
    $(y_1(j), y_1'(j), y_1''(j)) \leftarrow \mathcal{RK}_1(j-\varepsilon)$
    $(y_2(j), y_2'(j), y_2''(j)) \leftarrow \mathcal{RK}_2(j-\varepsilon)$
    $k \leftarrow \arg\min_i\{y_i'(j) | i \in \{1,2\}\}$
    $(y(j), y'(j), y''(j)) \leftarrow (y_k(j), y_k'(j), y_k''(j))$
    $\mathsf{REV}(C) += \Delta\mathsf{REV}(j)$
  end for
  if $\overline{v}^V \geq \overline{v}^U - \frac{y(\overline{v}^U)}{y'(\overline{v}^U)}$ **then**
    $\mathsf{REV}(C) += \int_{\overline{v}^U - \frac{y(\overline{v}^U)}{y'(\overline{v}^U)}}^{\overline{v}^V} \overline{x}^{V*}(v)\cdot h^V(v)\mathrm{d}v$
  end if
end for
return The C^* with the highest $\mathsf{REV}(C)$

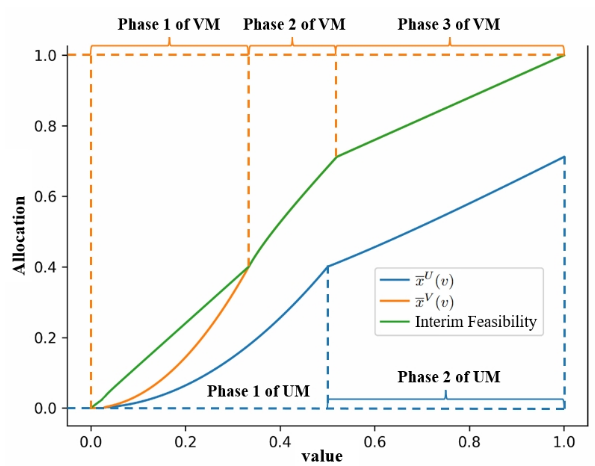

Figure 1: Optimal allocation when $n = 2, q = 0.4$, the value of UMs and VMs both follow uniform distribution on $[0, 1]$

Future Directions

This work investigates optimal auction design for mixed bidders. We characterize allocation and payment rules under IR and IC constraints and design optimal auctions for single bidders and optimal lookahead auctions for multiple bidders.

It remains unknown whether the optimal auction in the multi-bidder case is a lookahead auction. Given their ease of implementation and proven optimality for UMs with iid values, we conjecture that it is also optimal for the mixed bidders with iid classes.

Another future direction is applying the 'simple versus optimal' paradigm to our settings. While our derived optimal mechanism is complex and randomized, simpler alternatives like second-price auctions with reserves may provide constant approximations to the optimal for mixed-bidder auctions.

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A brief introduction to the single parameter environment

Suppose there are n bidders which are utility maximizers (UM). Each bidder has a private value v_i and they are required to bid their value. After collecting the bids $\boldsymbol{b} = (b_i)_{i\in[n]}$, the mechanism determine the allocation $\boldsymbol{x}(\boldsymbol{b}) = (x_i)_{i\in[n]}$ and the payment $\boldsymbol{p}(\boldsymbol{b})_{i\in[n]}$. Suppose the i-th bidder's value follows $\mathcal{F}_i$ whose pdf is $f_i(\cdot)$ and we assume it is regular.

Let $\boldsymbol{v}_{-i}$ be all bidders' value except the i-th one. Denote $\overline{x}_i(v_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i(v_i, \boldsymbol{v}_{-i})]$ and $\overline{p}_i(v_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[p_i(v_i, \boldsymbol{v}_{-i})]$ as the interim allocation and payment for the i-th bidder. The Myerson's lemma characterizes the unique interim payment function such that the IR and IC constraints when fixing $\overline{x}_i(\cdot)$.

Lemma 6. *Suppose the expected utility when bidding 0 is also 0. Then $(\overline{x}_i(\cdot), \overline{p}_i(\cdot))$ satisfies the IR and IC constraints if and only if:*

- *$\overline{x}_i(\cdot)$ is monotone increasing.*
- $\forall v, \quad \overline{p}_i(v) = v \cdot \overline{x}_i(v) - \int_0^v \overline{x}_i(u)\mathrm{d}u.$

Myerson's lemma is important for several reasons:

- It indicates that every monotone allocation can be IR and IC with a corresponding payment function.
- We can express our objective function only with $\overline{x}_i(\cdot)$ rather than $(\overline{x}_i(\cdot), \overline{p}_i(\cdot))$. It will be more convenient to further optimize.
- It does not assume the sum of all bidders' allocations cannot be larger than 1. Thus, it also works for position auctions or digital goods auctions.

Define the virtual value of the i-th bidder as $\varphi_i(v) = v - \frac{1-\mathcal{F}_i(v)}{f_i(v)}$. The following lemma shows that using Myerson's payment, the expected revenue is equal to the expected virtual value:

Lemma 7 (Revenue equivalence). *Using the payment function in lemma 6, we have*

$$\mathbb{E}[\overline{p}_i(v)] = \mathbb{E}[\overline{x}_i(v) \cdot \varphi(v)].$$

According to lemma 7, the virtual value can be regarded as 'revenue that can be extracted'. The following theorem is derived from the two lemmas above which characterizes the optimal single-item auction:

Theorem 2 (Myerson's auction (Myerson 1981)). *The optimal single-item auction for bidders with regular value distributions is to allocate the item to the bidder with the highest positive virtual value.*

This theorem can be extended to the non-regular case with an 'ironing' process, which we omit here. As corollaries of theorem 2, for bidders with i.i.d. regular distributions or a single bidder, Myerson's auction degenerates to simple auction formats as follows:

Corollary 1. *For bidders with i.i.d. regular value distributions, the optimal single-item auction is a second-price auction with reserve. That is, set a reserve price p such that $\varphi(p) = 0$, then*

- *If at least two bidders' values are larger than p, then the item is allocated to the bidder with the highest value. The payment is the value of the second-highest bidder.*
- *If only one bidder's value is larger than p, then allocate the item to her. The payment is p.*
- *If no one has value larger than p, do not allocate the item.*

Corollary 2. *For a single bidder, the optimal single-item auction is a posted price auction. That is, set a posted price p such that $\varphi(p) = 0$. If the bidder's value is larger than p, then allocate the item to her, the payment is p. Otherwise do not allocate the item.*

Although Myerson's approach is well-developed and has a wide range of applications, it cannot be applied in our multi-parameter setting. The main difficulty is that the IC constraints cannot be guaranteed only by a delicate payment function but also by the allocation function for both types of bidders. Moreover, the coupling of the allocation leads to no succinct expression of the revenue as in lemma 7. The optimal allocation is not a dichotomy like Myerson's auction, that is the reason why we use the variational method to optimize it.

Failure of IR and IC single-parameter mechanism

A single-parameter mechanism here means that the bidders only bid their value rather than their classes. We show such mechanisms cannot obtain a reasonable revenue for mixed bidders especially when the bidders are more likely VMs.

For the single parameter setting, the optimal IR and IC auction design follows directly from Myerson's work (Myerson 1981). Specifically, for the i-th bidder, now her value follows a value distribution denoted by $\mathcal{F}^T$, whose pdf is f^T, as the mixture distribution over $\mathcal{F}^U$ when being a UM and $\mathcal{F}^V$ when being a VM. Denote $\overline{x}_i(v) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i(v, \boldsymbol{v}_{-i})]$ and $\overline{p}_i(v) = \mathbb{E}_{\boldsymbol{v}_{-i}}[p_i(v, \boldsymbol{v}_{-i})]$ as the interim allocation and payment for the i-th bidder respectively. To satisfy the IC constraints for the case the i-th bidder is a UM, it requires:

$$\forall \hat{b}, v_i, \quad v_i \cdot \overline{x}_i(b) - \overline{p}_i(b) \leq v_i \cdot \overline{x}_i(v_i) - \overline{p}_i(v_i).$$

By the Myerson's lemma, for given allocation $\overline{x}_i(\cdot)$, there is unique payment $\overline{p}_i(\cdot)$ satisfying IC constraints:

Lemma 8 (Myerson's lemma (Myerson 1981)). *For given allocation rule $\overline{x}_i(\cdot)$, the only payment rule satisfying the IC constraints is:*

$$\forall v_i, \quad \overline{p}_i(v_i) = v_i \cdot \overline{x}_i(v_i) - \int_0^{v_i} \overline{x}_i(u)\mathrm{d}u.$$

This means the obtained revenue of an IR and IC mechanism does not depend on whether a bidder is UM or VM. Following the same derivation by (Myerson 1981), we have revenue equivalence. Formally, denote the virtual value by

$$\varphi_T(v) = v - \frac{1 - \mathcal{F}^T(v)}{f^T(v)},$$

then

$$\mathbb{E}_{\boldsymbol{v}}[\sum_{i=1}^n p_i(\boldsymbol{v})] = \mathbb{E}_{\boldsymbol{v}}[\sum_{i=1}^n x_i(\boldsymbol{v}) \cdot \varphi_T(v_i)].$$

Thus, the revenue maximized auction is to allocate the item to the bidder with the highest positive virtual value. The corresponding payment is

$$p_i(\boldsymbol{v}) = \begin{cases} \frac{\overline{p}_i(v_i)}{\overline{x}_i(v_i)} = v_i - \frac{\int_0^{v_i} \overline{x}_i(u)\mathrm{d}u}{\overline{x}_i(v_i)} & \text{if } i \text{ wins,} \\ 0 & \text{otherwise.} \end{cases}$$

Now we provide an example showing a single-parameter mechanism cannot lead to a reasonable revenue. Consider there is only one bidder who is a VM with probability 1. Her value follows the equal-revenue distribution $\mathcal{F}_{ER}^V$, that is:

$$\mathcal{F}_{ER}^V(v) = \begin{cases} 1 - \frac{1}{v} & \text{for } 1 \le v < c, \\ 1 & \text{for } v \ge c. \end{cases}$$

where $c > 1$ is a parameter to be determined. According to corollary 2, the optimal auction format should be a posted-price auction However, the basic property of the equal-revenue distribution is that, if you provide a posted price between 1 and c to a single buyer, the revenue will always be $v \cdot (1 - \mathcal{F}^V(v)) = 1$. Thus, the revenue of the optimal IR and IC auction is 1.

To be compared, consider the following multi-parameter mechanism: always selling the item to the bidder if her bid b satisfies $1 \le b \le c$, and the corresponding payment $p(b) = 1$ if the type she bids is UM, while $p(b) = b$ if the type she bids is VM. It is straightforward that this mechanism satisfies IR constraints. For the case that the bidder is a VM, deviating to bid her type as UM or bid other value does not increase the probability of winning the item, so as her utility. For the case that the bidder is a UM, deviating to bid her type as VM or bid other value does not decrease her payment, so as her utility. Thus, this mechanism also satisfies IC constraints. The expected revenue of this mechanism is

$$\mathbb{E}_{V \sim \mathcal{F}_{ER}^V}[V] = 1 + \ln c,$$

which can be arbitrarily larger than 1 for $c \to +\infty$.

Proof of Lemma 2

Property 1: $g_i(\cdot)$ is monotone increasing because

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}v} g_i(v) &= \frac{\mathrm{d}}{\mathrm{d}v}\Big(v - \frac{\int_0^v \overline{x}_i^U(u)\mathrm{d}u}{\overline{x}_i^U(v)}\Big) \\ &= \frac{\int_0^v \overline{x}_i^U(u)\mathrm{d}u \cdot \overline{x}_i^{U\prime}(v)}{(\overline{x}_i^U(v))^2} \ge 0. \end{aligned} \tag{11}$$

Property 2: If $\overline{x}_i^U(v_1) = \overline{x}_i^U(v_2)$, then

$$\begin{aligned} g_i(v_1) &= v_1 - \frac{\int_0^{v_1} \overline{x}_i^U(u)\mathrm{d}u}{\overline{x}_i^U(v_1)} \\ &= v_1 + (v_2 - v_1) - \frac{\int_0^{v_1} \overline{x}_i^U(u)\mathrm{d}u + \int_{v_1}^{v_2} \overline{x}_i^U(u)\mathrm{d}u}{\overline{x}_i^U(v_2)} \\ &= g_i(v_2). \end{aligned}$$

On the other hand, if $g_i(v_1) = g_i(v_2)$ but $\overline{x}_i^U(v_1) < \overline{x}_i^U(v_2)$, we have:

$$\begin{aligned} v_2 - v_1 &= \frac{\int_0^{v_2} \overline{x}_i^U(u)\mathrm{d}u}{\overline{x}_i^U(v_2)} - \frac{\int_0^{v_1} \overline{x}_i^U(u)\mathrm{d}u}{\overline{x}_i^U(v_1)} \\ &= \int_0^{v_1} \overline{x}_i^U(v)\mathrm{d}u \cdot \Big(\frac{1}{\overline{x}_i^U(v_2)} - \frac{1}{\overline{x}_i^U(v_1)}\Big) + \frac{\int_{v_1}^{v_2} \overline{x}_i^U(u)\mathrm{d}u}{\overline{x}_i^U(v_2)} \\ &< 0 + (v_2 - v_1), \end{aligned}$$

which is a contradiction. The first equality holds as $g_i(v_1) = g_i(v_2)$ and the last inequality holds as $\overline{x}_i^U(v_1) < \overline{x}_i^U(v_2)$ and the monotonicity of $\overline{x}_i^U(\cdot)$.

Property 3: It is straightforward since $\int_0^v \overline{x}_i^U(u)\mathrm{d}u$ is continuous w.r.t. v.

Property 4: We have:

$$\ln(\overline{x}_i^U(v))' = \frac{g_i'(v)}{v - g_i(v)},$$

which leads to the property.

Case study: Uniform distribution

We provide the optimal mechanism in closed form when the distributions of UM and VM are both uniform distributions. Suppose the support of the distributions are $[0, \overline{v}^U]$ and $[0, \overline{v}^V]$ for UM and VM respectively. Then

$$\begin{aligned} h^U(v) &= q \cdot \varphi^U(v) f^U(v) = q\Big(\frac{2v}{\overline{v}^U} - 1\Big), \\ h^{U\prime}(v) &= \frac{2q}{\overline{v}^U}, h^V(v) = (1-q) \cdot v \cdot f^V(v) = \frac{1-q}{\overline{v}^V} \cdot v. \end{aligned}$$

Substituting them into ODE (6) and denote $\alpha = \frac{2q\overline{v}^V}{(1-q)\overline{v}^U}$ as a constant, we have

$$\frac{g'(v) \cdot v}{v - g(v)} = \alpha, \tag{12}$$

which is homogeneous. Let $z(v) = \frac{g(v)}{v}$, then

$$\frac{z}{\alpha - \alpha \cdot z - z^2}\mathrm{d}z = \frac{1}{v}\mathrm{d}v.$$

By taking integral on both sides, we have

$$v \cdot (\phi_1 - z)^{\frac{1}{2} - \sqrt{\frac{\alpha}{\alpha+4}}} \cdot (z - \phi_2)^{\frac{1}{2} + \sqrt{\frac{\alpha}{\alpha+4}}} = D, \tag{13}$$

where D is a constant, and

$$\phi_1 = \frac{-\alpha + \sqrt{\alpha(\alpha+4)}}{2}, \phi_2 = \frac{-\alpha - \sqrt{\alpha(\alpha+4)}}{2},$$

which are the two roots of equation $\alpha - \alpha \cdot z - z^2 = 0$.

Since $\phi_2 \le z \le \phi_1$, we have

$$\text{LHS of Eqn. (13)} \le v \cdot (\phi_1 - \phi_2).$$

Let $x \to 0$, the LHS of Eqn. (13) also $\to 0$. Thus, the parameter D must be 0. Since $\phi_2 < 0 < \phi_1$, we have $z(v) = \phi_1, g(v) = \phi_1 \cdot v$. Substituting into Eqn. (7), we have

$$\overline{x}^U(v) = \Big(\frac{v}{\overline{v}}\Big)^{\frac{\alpha + \sqrt{\alpha(\alpha+4)}}{2}}.$$

Let $\phi = \frac{\alpha+\sqrt{\alpha(\alpha+4)}}{2}$ and take the derivative of the RHS of Eqn. (2) with respect to $\overline{v}$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\overline{v}}\mathsf{REV}(\mathcal{M}) \\
&= \int_0^{\overline{v}} \left(\frac{\mathrm{d}}{\mathrm{d}\overline{v}}\left(\frac{v}{\overline{v}}\right)^{\phi}\right)(h^U(v) + g'(v)h^V(g(v)))\mathrm{d}v \\
&= -\phi \cdot \overline{v}^{-\phi-1} \cdot \int_0^{\overline{v}} \left(\left(\frac{2q}{\overline{v}^U} + \frac{(1-q)\phi_1^2}{\overline{v}^V}\right)v^{\phi+1} - q \cdot v^{\phi}\right)\mathrm{d}v \\
&= -\phi \cdot \overline{v}^{-\phi-1} \cdot \left(\frac{\frac{2q}{\overline{v}^U} + \frac{(1-q)\phi_1^2}{\overline{v}^V}}{\phi+2} \cdot v^{\phi+2} - \frac{q}{\phi+1} \cdot v^{\phi+1}\right)\Bigg|_{v=0}^{\overline{v}} \\
&= -\phi \cdot \left(\frac{\frac{2q}{\overline{v}^U} + \frac{(1-q)\phi_1^2}{\overline{v}^V}}{\phi+2} \cdot \overline{v} - \frac{q}{\phi+1}\right).
\end{aligned}
$$

Let $\phi = \frac{\alpha+\sqrt{\alpha(\alpha+4)}}{2}$ and take the derivative of the RHS of Eqn. (2) with respect to $\overline{v}^U$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}\overline{v}^U}\mathsf{REV}(\mathcal{M}) = -\phi \cdot \left(\frac{\frac{2q}{\overline{v}^U} + \frac{(1-q)\phi_1^2}{\overline{v}^V}}{\phi+2} \cdot \overline{v}^U - \frac{q}{\phi+1}\right),
$$

which is a linear function with respect to $\overline{v}$. Let $\frac{\mathrm{d}}{\mathrm{d}\overline{v}}\mathsf{REV}(\mathcal{M}) = 0$, we have the optimal $\overline{v}$ is that:

$$
\overline{v} = \frac{q(\phi+2)}{(\phi+1)\left(\frac{2q}{\overline{v}^U} + \frac{(1-q)\phi_1^2}{\overline{v}^V}\right)} = \frac{\overline{v}^U}{2}.
$$

To sum up, we derive the optimal mechanism for a single bidder with value following uniform distributions:

$$
\overline{x}^U(v) = \begin{cases} (\frac{2v}{\overline{v}^U})^{\phi} & 0 < v \le \frac{\overline{v}^U}{2}, \\ 1 & v > \frac{\overline{v}^U}{2}. \end{cases}
$$

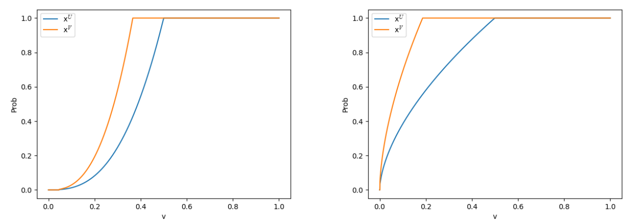
$$
\overline{x}^V(v) = \begin{cases} (\frac{2v}{\phi_1 \cdot \overline{v}^U})^{\phi} & 0 < v \le \frac{\phi_1 \cdot \overline{v}^U}{2}, \\ 1 & v > \frac{\phi_1 \cdot \overline{v}^U}{2}. \end{cases}
$$

$$
\overline{p}^U(v) = \begin{cases} \phi_1 \cdot v \cdot (\frac{2v}{\overline{v}^U})^{\phi} & 0 < v \le \frac{\overline{v}^U}{2}, \\ \frac{\phi_1 \overline{v}^U}{2} & v > \frac{\overline{v}^U}{2}. \end{cases}
$$

$$
\overline{p}^V(v) = \begin{cases} v \cdot (\frac{2v}{\phi_1 \cdot \overline{v}^U})^{\phi} & 0 < v \le \frac{\phi_1 \cdot \overline{v}^U}{2}, \\ v & v > \frac{\phi_1 \cdot \overline{v}^U}{2}. \end{cases}
$$

where $\alpha = \frac{2q\overline{v}^U}{(1-q)\overline{v}^U}$, $\phi_1 = \frac{-\alpha+\sqrt{\alpha(\alpha+4)}}{2}$, $\phi = \frac{\alpha+\sqrt{\alpha(\alpha+4)}}{2}$ are constants. This mechanism can be implemented by the following process: when the bidder bids $\theta = (t, v)$, we allocate the item to her with probability $\overline{x}^t(v)$. The payment is $\frac{\overline{p}^t(v)}{\overline{x}^t(v)}$ if she is allocated, otherwise the payment is 0.

Remark 1. *The Myerson auction for a single UM is to allocate the item to the bidder if and only if her value is larger than $\overline{v}$, such that the virtual value $\varphi^U(\overline{v}) = 0$. If the bidder's value follows a uniform distribution on $[0, \overline{v}^U]$, $\varphi^U(v) = 2v - \overline{v}^U, \forall v$. Thus $\overline{v} = \frac{\overline{v}^U}{2}$ which is the $\overline{v}$ in our mechanism. It can be observed that even when $q < 1$, the optimal mechanism allocates to UM with a positive virtual value with probability 1. If the UM's virtual value is negative, then the probability that she is allocated is strictly smaller than 1.*



(a) $p = 0.5$ (b) $p = 0.1$

The following figure illustrates the optimal allocation for both types when $\mathcal{F}^U$ and $\mathcal{F}^V$ are uniform distributions over $[0, 1]$, with $q = 0.5$ and $q = 0.1$, respectively.

Proof of Lemma 5

Proof: Necessity is obvious; we prove sufficiency. An interim allocation satisfying the condition can be implemented as follows: After collecting all bids, identify the highest-scoring bidder i. Allocate the item to i with probability $\overline{x}^t(v)/\Pr[i \text{ has the highest score} \mid \theta]$. $\square$